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SOME LIMITS FOR THE LAPLACE TRANSFORM OF THE BROWNIAN MOTION'S FIRST HIT TO A LINEAR FUNCTION

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Abstract. The aim of this paper is to examine some limits related to the Laplace transform of the Brownian motion before its first hit to a linear boundary.

1. Introduction. The importance of the Brownian motion is due to its wide use in many scientific areas as well as its large applicability in different practical tasks. There are many studies devoted to the problem of its first hit to some boundary – see for example [8, 3, 6, 9] Also, some two-sided tasks are discussed in [1, 7, 10].

We are interested in some limits of the form $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T}\right]$ restricted on the sample paths at which the Brownian motion still does not hit a linear boundary. This term is related to the moment generating function of the normal

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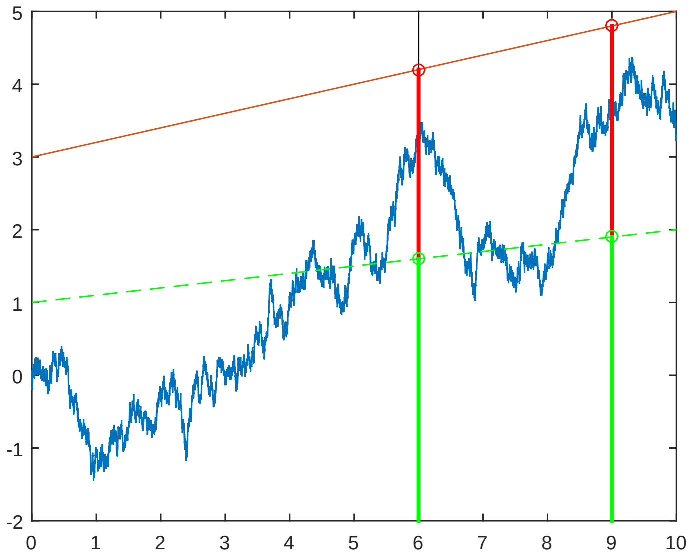

Fig. 1. Illustration of the problem

distribution. Also, we consider an additional restriction only at the moment T by another linear function. We can proceed in two manners using the symmetry of the Brownian motion. First, we may examine only upper-hitting tasks and arbitrary constant θ. Alternatively, we may assume that $\theta > 0$ and consider both lower and upper hits. We chose the second scheme. We illustrate our task for an upper-hitting problem by Figure 1. A trajectory that stays below the boundary $b(t) = b_1 t + b_2$ (red line) till moment $T = 10$ is plotted by blue color. Thus, we take the expectation over all such sample paths and search for the limit $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{B_t<b(t),\ t\in[0,T]}\right]$. In addition, let $z(t)$ be another linear function that is below $b(t)$ for large enough values of t – we plot it by a green dashed line. The second our task is to obtain the limit imposing the additional condition that the value of B_T is larger than $z(T)$, i.e. to fall in the red part. Thus the limit we search turns into $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{B_t<b(t)\ \forall\ t\in[0,T],\ B_T>z(T)}\right]$. Note that we do not restrict the sample paths to stay in the strip between the green and red lines but only the Brownian motion's value at the terminal moment T to be above $z(T)$ (of course below $b(T)$ too).

The derived results are inspired by the area of financial mathematics.

The famous Black and Scholes [2] model presents a financial asset as a geometric Brownian motion and thus the term $e^{\theta B_T}$ arises. In this light, the obtained limits are useful for evaluating different American-style financial instruments as well as the popular barrier options – European or American. More precisely, these results are important for the so-called perpetual derivatives which are written without maturity constraints – mathematically, this means $T = \infty$. For example, in [12] are considered some American-style futures contracts. It turns out that the early exercise is always optimal for a short-positioned futures if the underlying asset does not pay dividends – this means that the optimal boundary is infinitely large. On the other hand, if the dividend rate is positive, then the optimal boundary exists and it tends to a finite value even though the dividend tends to zero. This discontinuity appears because a term of the form $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{B_t<b(t),\ t\in[0,T]}\right]$ has a discontinuity at the point $k = -\dfrac{\theta^2}{2}$. Other similar applications can be found in [11] where some power payoffs are considered. Last but not least, the results for the limit $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{B_t<b(t)\ \forall\ t\in[0,T],\ B_T>z(T)}\right]$ are important for option pricing since the underlying asset is restricted by the strike. Thus the function $z(t)$ arises.

2. Some lemmas. Let B_t be a Brownian motion under the natural filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$. Let ζ be the first hitting moment of the Brownian motion to the linear function $b(t) = b_1 t + b_2$. We shall denote by $N(\cdot)$ the cumulative distribution function of the standard normal distribution. We need the following theorem whose proof can be found in [9, Theorem 3.2].

Theorem 2.1. *If b_1 and θ are arbitrary real numbers, $b_2 > 0$, and $z < b(T)$, then*

(2.1)

$$V(\theta, z, T; b_1, b_2) \equiv \mathbb{E}\left[e^{\theta B_T} I_{B_T>z,\zeta>T}\right] =$$

$$= \exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} N\left(\dfrac{b(T)-T\theta}{\sqrt{T}}\right) - N\left(\dfrac{z-T\theta}{\sqrt{T}}\right) \\ +e^{2b_2(\theta-b_1)}\left(N\left(\dfrac{z-T\theta-2b_2}{\sqrt{T}}\right) - N\left(\dfrac{b(T)-T\theta-2b_2}{\sqrt{T}}\right)\right)\end{array}\right].$$

If $z > b(T)$ and $b_2 < 0$, then the symmetry of the Brownian motion leads *to*

(2.2)
$$V(\theta, z, T; b(\cdot)) \equiv \mathbb{E}\left[e^{\theta B_T} I_{B_T<z,\zeta>T}\right] =$$

$$= \exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} N\left(\frac{z-T\theta}{\sqrt{T}}\right) - N\left(\frac{b(T)-T\theta}{\sqrt{T}}\right) \\ -e^{2b_2(\theta-b_1)}\left(N\left(\frac{z-T\theta-2b_2}{\sqrt{T}}\right) - N\left(\frac{b(T)-T\theta-2b_2}{\sqrt{T}}\right)\right) \end{array}\right].$$

P r o o f. The proof of formula (2.1) can be found in [9, Theorem 3.2]. Let us turn to formula (2.2). Let a new Brownian motion be defined as the opposite of the original one, $\overline{B}_t = -B_t$. The stopping time ζ can be viewed as the first hit of $\overline{B}_t$ to the function $\overline{b}(t) = -b_1 t - b_2$. We prove the desired result using formula (2.1) and having in mind $E\left[e^{\theta B_T} I_{B_T<z,\zeta>T}\right] = E\left[e^{-\theta \overline{B}_T} I_{\overline{B}_T>-z,\zeta>T}\right]$. Note that formula (2.1) holds for an arbitrary real number θ and $N(-x) = 1 - N(x)$. □

In the first case ($b_2 > 0$) the boundary is above the initial point of the Brownian motion, whereas it is below when $b_2 < 0$.

Corollary 2.1. *Let $z(t) = z_1 t + z_2$ be another linear function, that satisfies the conditions:*

1. *If $b_2 > 0$, then $z_1 \le b_1$. In addition, if $z_1 = b_1$, then $z_2 < b_2$.*
2. *If $b_2 < 0$, then $z_1 \ge b_1$. In addition, if $z_1 = b_1$, then $z_2 > b_2$.*

Expectation (2.1) *for $z = z(T)$ can be rewritten as*

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T>z(T),\zeta>T}\right] = \frac{e^{\left(-\frac{b_1^2}{2}+b_1\theta\right)T}}{\sqrt{2\pi T}} \int\limits_{(z_1-b_1)T+z_2}^{b_2} e^{-\frac{y^2}{2T}} e^{-y(b_1-\theta)}\left(1 - e^{\frac{2b_2(y-b_2)}{T}}\right) dy.$$

Note that if $z_1 = b_1$, then

$$0 < \lim_{T\to\infty} T \int\limits_{z_2}^{b_2} e^{-\frac{y^2}{2T}} e^{-y(b_1-\theta)}\left(1 - e^{\frac{2b_2(y-b_2)}{T}}\right) dy < \infty. \tag{2.3}$$

Proof. We have

$$e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] \tag{2.4}$$

$$= e^{\left(k+\frac{\theta^2}{2}\right)T}\left[N\left((b_1-\theta)\sqrt{T}+\frac{b_2}{\sqrt{T}}\right)-N\left((z_1-\theta)\sqrt{T}+\frac{z_2}{\sqrt{T}}\right)\right]$$

$$-e^{\left(k+\frac{\theta^2}{2}\right)T}e^{2b_2(\theta-b_1)}\left[N\left((b_1-\theta)\sqrt{T}-\frac{b_2}{\sqrt{T}}\right)-N\left((z_1-\theta)\sqrt{T}+\frac{z_2-2b_2}{\sqrt{T}}\right)\right].$$

Using the change of variables

$$y = x\sqrt{T}-m_1T \Leftrightarrow x = m_1\sqrt{T}+\frac{y}{\sqrt{T}}, \tag{2.5}$$

for $m_1 = b_1-\theta$, we derive for the first term

$$e^{\left(k+\frac{\theta^2}{2}\right)T}\left[N\left((b_1-\theta)\sqrt{T}+\frac{b_2}{\sqrt{T}}\right)-N\left((z_1-\theta)\sqrt{T}+\frac{z_2}{\sqrt{T}}\right)\right]$$

$$= \frac{e^{\left(k+\frac{\theta^2}{2}\right)T}}{\sqrt{2\pi}}\int\limits_{(z_1-\theta)\sqrt{T}+\frac{z_2}{\sqrt{T}}}^{(b_1-\theta)\sqrt{T}+\frac{b_2}{\sqrt{T}}} e^{-\frac{x^2}{2}}dx$$

$$= \frac{e^{\left(k+\frac{\theta^2}{2}\right)T}}{\sqrt{2\pi T}}\int\limits_{(z_1-b_1)T+z_2}^{b_2} e^{-\frac{1}{2}\left((b_1-\theta)\sqrt{T}+\frac{y}{\sqrt{T}}\right)^2}dy \tag{2.6}$$

$$= \frac{e^{\left(k-\frac{b_1^2}{2}+b_1\theta\right)T}}{\sqrt{2\pi T}}\int\limits_{(z_1-b_1)T+z_2}^{b_2} e^{-\frac{y^2}{2T}}e^{-y(b_1-\theta)}dy.$$

Let us turn to the second part of formula (2.4). The change of variables $u = y+2b_2$ in addition to the one used above leads to

$$e^{\left(k+\frac{\theta^2}{2}\right)T}e^{2b_2(\theta-b_1)}\left[N\left((b_1-\theta)\sqrt{T}-\frac{b_2}{\sqrt{T}}\right)-N\left((z_1-\theta)\sqrt{T}+\frac{z_2-2b_2}{\sqrt{T}}\right)\right]$$

$$= \frac{e^{\left(k+\frac{\theta^2}{2}\right)T}e^{2b_2(\theta-b_1)}}{\sqrt{2\pi T}}\int\limits_{(z_1-b_1)T+z_2-2b_2}^{-b_2} e^{-\frac{1}{2}\left((b_1-\theta)\sqrt{T}+\frac{y}{\sqrt{T}}\right)^2}dy \tag{2.7}$$

$$= \frac{e^{\left(k-\frac{b_1^2}{2}+b_1\theta\right)T}}{\sqrt{2\pi T}}\int\limits_{(z_1-b_1)T+z_2}^{b_2} e^{-\frac{u^2}{2T}}e^{-u(b_1-\theta)}e^{\frac{2b_2(u-b_2)}{T}}du.$$

Combining equations (2.6) and (2.7), we derive

$$e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right]$$
$$= \frac{e^{\left(k-\frac{b_1^2}{2}+b_1\theta\right)T}}{\sqrt{2\pi T}} \int\limits_{(z_1-b_1)T+z_2}^{b_2} e^{-\frac{y^2}{2T}} e^{-y(b_1-\theta)}\left(1-e^{\frac{2b_2(y-b_2)}{T}}\right) dy.$$

Restrictions (2.3) hold because $\left(1-e^{\frac{2b_2(y-b_2)}{T}}\right)$ tends to zero as $\dfrac{2b_2\left(b_2-y\right)}{T}$ for $T\to\infty$ and the term $e^{-\frac{y}{2T}} \to 1$ in the integral domain which is finite when $z_1 = b_1$. □

Suppose now that the conditions of Corollary 2.1 hold, $z_1 < b_1$, and $z_2 > b_2$. Let us denote by $\bar{t}$ the root of $b\left(t\right) = z\left(t\right)$, i.e. $\bar{t} = \dfrac{z_2 - b_2}{b_1 - z_1}$. We can rewrite expectation (2.1) for $T > \bar{t}$ as

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T>z(T),\zeta>T}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{\theta B_T} I_{B_T>z(T),\zeta>T}\middle| \mathcal{F}_{\bar{t}}\right]\right] \tag{2.8}$$
$$= \int\limits_{-\infty}^{b(\bar{t})} \mathbb{E}^{\bar{t},u}\left[e^{\theta B_T} I_{B_T>z(T),\zeta>T}\right] d\mathbb{P}\left(B_{\bar{t}} < y\right).$$

The notation $\mathbb{E}^{\bar{t},u}$ above means the expectation under the assumption that the Brownian motion has a value u at the moment $\bar{t}$.

We shall prove now several lemmas – later we shall use them to prove our main results.

Lemma 2.1. *If the constant m is positive, then*

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{m^2}{2}} \frac{m}{m^2+1} < N\left(-m\right) < e^{\frac{-m^2}{2}} \frac{1}{m\sqrt{2\pi}}.$$

P r o o f. The lemma follows a well-known result for the Mills ratio for $x > 0$, proven in [5],

$$\frac{x}{x^2+1} \le e^{\frac{x^2}{2}} \int\limits_x^{\infty} e^{-\frac{u^2}{2}} du \le \frac{1}{x}. \qquad \square$$

Lemma 2.2. *If the constants m_1 and m_2 are positive and $T > \dfrac{m_2}{m_1}$, then the following inequalities hold*

$$
\begin{aligned}
&\sqrt{\frac{2T}{\pi}}\frac{m_2 e^{m_1 m_2}}{m_1^2T^2-m_2^2}\left[1-T\frac{3m_1^2T^2+m_2^2}{\left(m_1^2T^2-m_2^2\right)^2}\right]\exp\left(-\frac{1}{2}\left(m_1^2T+\frac{m_2^2}{T}\right)\right)\\
&< N\left(-m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-e^{2m_1m_2}N\left(-m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\\
&< \sqrt{\frac{2T}{\pi}}\frac{m_2 e^{m_1 m_2}}{m_1^2T^2-m_2^2}\exp\left(-\frac{1}{2}\left(m_1^2T+\frac{m_2^2}{T}\right)\right).
\end{aligned} \tag{2.9}
$$

Note that the first term is positive for large enough values of T.

Proof. We have

$$
\begin{aligned}
&N\left(-m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-e^{2m_1m_2}N\left(-m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\\
&=\frac{1}{\sqrt{2\pi}}\left[\int_{m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}}^{\infty} e^{-\frac{x^2}{2}}\,dx-e^{2m_1m_2}\int_{m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}}^{\infty} e^{-\frac{x^2}{2}}\,dx\right]\\
&=\frac{1}{\sqrt{2\pi}}\left[\int_{\sqrt{T}}^{\infty} e^{-\frac{\left(m_1y-\frac{m_2}{y}\right)^2}{2}}\left(m_1+\frac{m_2}{y^2}\right)dy\right.\\
&\qquad\left.-e^{2m_1m_2}\int_{\sqrt{T}}^{\infty} e^{-\frac{\left(m_1y+\frac{m_2}{y}\right)^2}{2}}\left(m_1-\frac{m_2}{y^2}\right)dy\right]
\end{aligned} \tag{2.10}
$$

$$
\begin{aligned}
&=\sqrt{\frac{2}{\pi}}e^{m_1m_2}m_2\int_{\sqrt{T}}^{\infty}\frac{e^{-\frac{m_1^2y^2+m_2^2y^{-2}}{2}}}{y^2}dy\\
&=\sqrt{\frac{2}{\pi}}e^{m_1m_2}m_2\int_{0}^{\frac{1}{\sqrt{T}}} e^{-\frac{m_1^2z^{-2}+m_2^2z^2}{2}}dz.
\end{aligned}
$$

We have used above the changes of variables

$$x = m_1 y - \frac{m_2}{y}$$
$$x = m_1 y + \frac{m_2}{y}$$
$$y = \frac{1}{z}.$$

Let M_1 and M_2 be defined as $M_1 = m_1^2$ and $M_2 = m_2^2$. We need the following derivatives before to continue

$$\begin{aligned}
&\left(-\frac{M_1 z^{-2} + M_2 z^2}{2}\right)' = \frac{M_1}{z^3} - M_2 z \\
&\left(\frac{z^3}{M_1 - M_2 z^4}\right)' = z^2 \frac{M_2 z^4 + 3M_1}{(M_1 - M_2 z^4)^2} \\
&\left(\left(3M_1 z^{-4} + M_2\right)\left(\frac{z^3}{M_1 - M_2 z^4}\right)^3\right)' \\
&\qquad = 3\left(\frac{z}{M_1 + M_2 z^4}\right)^4 \left(M_2^2 z^8 + 10 M_1 M_2 z^4 + 5M_1^2\right).
\end{aligned} \tag{2.11}$$

Applying twice integration by parts in formula (2.10) having in mind the restriction $T > \dfrac{m_2}{m_1}$, and using derivatives (2.11), we derive

$$\begin{aligned}
&\int_0^{\frac{1}{\sqrt{T}}} e^{-\frac{M_1 z^{-2} + M_2 z^2}{2}} dz = \int_0^{\frac{1}{\sqrt{T}}} \frac{z^3}{M_1 - M_2 z^4} d\left(e^{-\frac{M_1 z^{-2} + M_2 z^2}{2}}\right) \\
&= \frac{z^3}{M_1 - M_2 z^4} e^{-\frac{M_1 z^{-2} + M_2 z^2}{2}} \Bigg|_0^{\frac{1}{\sqrt{T}}} - \int_0^{\frac{1}{\sqrt{T}}} z^2 \frac{M_2 z^4 + 3M_1}{(M_1 - M_2 z^4)^2} e^{-\frac{M_1 z^{-2} + M_2 z^2}{2}} dz \\
&= \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}} - \int_0^{\frac{1}{\sqrt{T}}} z^2 \frac{M_2 z^4 + 3M_1}{(M_1 - M_2 z^4)^2} e^{-\frac{M_1 z^{-2} + M_2 z^2}{2}} dz \qquad (2.12) \\
&= \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}}
\end{aligned}$$

$$- \int_0^{\frac{1}{\sqrt{T}}} z^2 \frac{M_2 z^4 + 3M_1}{(M_1 - M_2 z^4)^2} \frac{z^3}{M_1 - M_2 z^4} d\left(e^{-\frac{M_1 z^{-2} + M_2 z^2}{2}}\right)$$

$$= \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}}$$

$$- \int_0^{\frac{1}{\sqrt{T}}} \left(3M_1 z^{-4} + M_2\right) \left(\frac{z^3}{M_1 - M_2 z^4}\right)^3 d\left(e^{-\frac{M_1 z^{-2} + M_2 z^2}{2}}\right)$$

$$= \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}}$$

$$- \left(3M_1 z^{-4} + M_2\right) \left(\frac{z^3}{M_1 - M_2 z^4}\right)^3 e^{-\frac{M_1 z^{-2} + M_2 z^2}{2}} \Bigg|_0^{\frac{1}{\sqrt{T}}}$$

$$+ 3 \int_0^{\frac{1}{\sqrt{T}}} e^{-\frac{M_1 z^{-2} + M_2 z^2}{2}} \left(\frac{z}{M_1 + M_2 z^4}\right)^4 \left(M_2^2 z^8 + 10 M_1 M_2 z^4 + 5 M_1^2\right) dz$$

$$> \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}} \left[1 - T \frac{3M_1 T^2 + M_2}{\left(M_1 T^2 - M_2\right)^2}\right]. \tag{2.13}$$

We prove the second inequality in (2.9) combining equations (2.10) and (2.12). The first one is a consequence of inequality (2.13). □

Lemma 2.3. *The following statements hold.*

1. *If* $k > 0$, $m < 0$, *and* $k > \frac{m^2}{2}$, *then* $\lim_{T \to \infty} e^{kT} N\left(m\sqrt{T}\right) = \infty$.
2. *If* $k > 0$, $m < 0$, *and* $k \le \frac{m^2}{2}$, *then* $\lim_{T \to \infty} e^{kT} N\left(m\sqrt{T}\right) = 0$.

P r o o f. These statements follow Lemma 2.1. □

Lemma 2.4. *If* $m_1 < 0$ *and* $m_2 > 0$, *then*

1. *If* $k \le \frac{m_1^2}{2}$, *then*

$$\lim_{T \to \infty} e^{kT} \left[N\left(m_1\sqrt{T} + \frac{m_2}{\sqrt{T}}\right) - e^{2m_1 m_2} N\left(m_1\sqrt{T} - \frac{m_2}{\sqrt{T}}\right)\right] = 0.$$

2. *If* $k > \frac{m_1^2}{2}$, *then*

$$\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-e^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right]=\infty.$$

Proof. The results follow Lemma 2.2. □

Lemma 2.5. *If* $m_1 < 0$, $m_2 > 0$, $k > \frac{m_1^2}{2}$, $C > 1$, *and* $D \geq 0$ *then*

$$\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}\right)\right.$$
$$\left.-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2-D}{\sqrt{T}}\right)\right]=-\infty. \tag{2.14}$$

Proof. We shall establish first the result for $D = 0$. We can rewrite equation (2.14) as

$$\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right]$$
$$=\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-e^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right]$$
$$-\lim_{T\to\infty} e^{kT}\left[(C-1)\,e^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right].$$

Lemmas 2.1 and 2.2 show that the terms above are constrained by terms of type $e^{\left(k-\frac{m_1^2}{2}\right)T}\frac{A_1}{T\sqrt{T}}$ and $e^{\left(k-\frac{m_1^2}{2}\right)T}\frac{A_2}{\sqrt{T}}$, respectively, for large enough values of T and some positive constants A_1 and A_2. Thus the whole limit behavior is

$$\frac{e^{\left(k-\frac{m_1^2}{2}\right)T}}{T\sqrt{T}}\left[A_1-(C-1)\,A_2T\right],$$

which tends to minus infinity since $C > 1$.

Suppose now that $D > 0$. We can rewrite equation (2.14) as

$$\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}\right)-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2-D}{\sqrt{T}}\right)\right]$$
$$=\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}\right)-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2-D}{\sqrt{T}}\right)\right]$$
$$\pm\left\{\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right]\right\}$$
$$=\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right]$$
$$+\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}\right)-N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)\right]$$
$$-\lim_{T\to\infty} Ce^{kT}e^{2m_1m_2}\left[N\left(m_1\sqrt{T}+\frac{D-m_2}{\sqrt{T}}\right)-N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right].$$

We shall denote by I_1, I_2, and I_3 the three components above. The limit of I_1 is minus infinity because we have already proved the lemma for $D = 0$. Let us consider the term I_2. Changing the variables as in (2.5), we derive

$$I_2 := N\left(m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}\right)-N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)$$
$$=\int\limits_{m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}}^{m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}} e^{-\frac{x^2}{2}}\,dx$$
$$=\frac{1}{\sqrt{T}}\int\limits_{m_2}^{m_2+D} e^{-\frac{\left(m_1\sqrt{T}+\frac{y}{\sqrt{T}}\right)^2}{2}}dy$$
$$=\frac{e^{-\frac{m_1^2}{2}T}}{\sqrt{T}}\int\limits_{m_2}^{m_2+D} e^{-\frac{y^2}{2T}}e^{-m_1y}dy.$$

Analogously, using in addition the change $u = y + 2m_2 \Leftrightarrow y = u - 2m_2$, we obtain for I_3

$$I_3 := e^{2m_1m_2}\left[N\left(m_1\sqrt{T} + \frac{D - m_2}{\sqrt{T}}\right) - N\left(m_1\sqrt{T} - \frac{m_2}{\sqrt{T}}\right)\right]$$

$$= e^{2m_1m_2}\frac{e^{-\frac{m_1^2}{2}T}}{\sqrt{T}}\int\limits_{-m_2}^{-m_2+D} e^{-\frac{y^2}{2T}}e^{-m_1y}dy$$

$$= \frac{e^{-\frac{m_1^2}{2}T}}{\sqrt{T}}\int\limits_{m_2}^{m_2+D} e^{-\frac{u^2}{2T}}e^{-m_1y}e^{-2\frac{m_2^2-um_2}{T}}du.$$

We can observe that $I_3 > I_2$ because $m_2 > 0$ and $u > m_2$. Having in mind that $C > 1$ and $I_3 > 0$, we conclude $I_2 - CI_3 < I_2 - I_3 < 0$. We finish the proof combining this inequality and the limit $\lim\limits_{T\to\infty} I_1 = -\infty$. □

3. Main results.

We shall obtain now the results for the expectation $\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta}\right]$.

Theorem 3.1. *Let θ be a positive number, $b(\cdot)$ be the linear function $b(t) = b_1t + b_2$, ζ be the Brownian motion's first hitting time to it, and k be an arbitrary constant. The following statements hold.*

1. *If* $\{b_2 = 0\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta}\right] = 0.$

2. *If* $\left\{b_2 \neq 0, k < -\dfrac{\theta^2}{2}\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta}\right] = 0.$

3. *If* $\left\{b_2 \neq 0, b_1 = \theta, k = -\dfrac{\theta^2}{2}\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta}\right] = 0.$

4. *If* $\left\{b_2 \neq 0, b_1 = \theta, k > -\dfrac{\theta^2}{2}\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta}\right] = \infty.$

5. *If* $\left\{b_2 > 0,\ b_1 < \theta,\ k = -\dfrac{\theta^2}{2}\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta}\right] = 0.$

6. *If* $\left\{b_2 > 0,\ b_1 > \theta,\ k = -\frac{\theta^2}{2}\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 1 - e^{2b_2(\theta-b_1)}$.

7. *If* $\left\{b_2 > 0,\ b_1 > \theta,\ k > -\frac{\theta^2}{2}\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.

8. *If* $\left\{b_2 > 0,\ b_1 < \theta,\ -\frac{\theta^2}{2} < k \le \frac{b_1^2}{2} - \theta b_1\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.

9. *If* $\left\{b_2 > 0,\ b_1 < \theta,\ \frac{b_1^2}{2} - \theta b_1 < k\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.

10. *If* $\left\{b_2 < 0,\ b_1 > \theta,\ k = -\frac{\theta^2}{2}\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.

11. *If* $\left\{b_2 < 0,\ b_1 < \theta,\ k = -\frac{\theta^2}{2}\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 1 - e^{2b_2(\theta-b_1)}$.

12. *If* $\left\{b_2 < 0,\ b_1 < \theta,\ k > -\frac{\theta^2}{2}\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.

13. *If* $\left\{b_2 < 0,\ b_1 > \theta,\ -\frac{\theta^2}{2} < k \le \frac{b_1^2}{2} - \theta b_1\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.

14. *If* $\left\{b_2 < 0,\ b_1 > \theta,\ \frac{b_1^2}{2} - \theta b_1 < k\right\}$, *then* $\lim\limits_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.

Proof. We shall discuss point by point all cases, paying special attention to a potential discontinuity at zero. If $b_2 > 0$, then formula (2.1), applied for $z = -\infty$, leads to

$$\begin{aligned} (3.1)\quad & \mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] \\ &= \exp\left(\frac{T\theta^2}{2}\right)\left[N\left(\frac{b(T) - T\theta}{\sqrt{T}}\right) - e^{2b_2(\theta-b_1)} N\left(\frac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right)\right]. \end{aligned}$$

Otherwise, if $b_2 < 0$, then formula (2.2), written for $z = \infty$, leads to

$$\begin{aligned} (3.2)\quad & \mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] \\ &= \exp\left(\frac{T\theta^2}{2}\right)\left[N\left(-\frac{b(T) - T\theta}{\sqrt{T}}\right) - e^{2b_2(\theta-b_1)} N\left(-\frac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right)\right]. \end{aligned}$$

1. The first statement holds because $\zeta = 0$ at every sample path when $b_2 = 0$. Note that the arguments of the normal CDFs in formulas (3.1) and (3.2) are equal, which confirms that the limit is zero.

2. If $\left\{b_2 \neq 0, k < -\frac{\theta^2}{2}\right\}$, then the limit is zero because the exponent in formulas (3.1) or (3.2) tends to zero and the other term is finite.

3. We see that both exponents in formulas (3.1) or (3.2) vanish, because $k = -\frac{\theta^2}{2}$ and $b_1 = \theta$. Also, the arguments of the normal CDFs tend to zero because $b_1 = \theta$.

4. Suppose that $\left\{b_1 = \theta, k > -\frac{\theta^2}{2}\right\}$ and $b_2 > 0$. We have indeterminacy of the kind infinity multiplied by zero. Notating $d = k + \frac{\theta^2}{2}$, using formula (3.1), and changing the variable $y = x\frac{\sqrt{T}}{b_2} \Leftrightarrow x = y\frac{b_2}{\sqrt{T}}$, we derive

$$\begin{aligned}
&e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] \\
&\quad = e^{T\left(k+\frac{\theta^2}{2}\right)}\left[N\left(\frac{b(T) - T\theta}{\sqrt{T}}\right) - e^{2b_2(\theta-b_1)}N\left(\frac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right)\right] \\
&\quad = e^{Td}\left[2N\left(\frac{b_2}{\sqrt{T}}\right) - 1\right] \\
&\quad = \frac{2e^{Td}}{\sqrt{2\pi}}\int_0^{\frac{b_2}{\sqrt{T}}} e^{-\frac{x^2}{2}}dx \\
&\quad = \frac{2e^{Td}}{\sqrt{2\pi T}}\int_0^1 e^{-\frac{x^2 b_2^2}{2T}}dx.
\end{aligned}$$

We can see that the integrand tends to one when $T \to \infty$ and thus the whole integral tends to one. This proves that the limit is infinite since $d > 0$. The statement can be proven in the same way when $b_2 < 0$ by the use of formula (3.2).

5. The limit is zero because the exponent in formula (3.1) vanishes and the arguments of the normal CDF tend to minus infinity.

6. This statement holds because the arguments of the normal CDF now tend to plus infinity.

7. The arguments of the normal CDF tend again to plus infinity and thus

$$N\left(\frac{b(T)-T\theta}{\sqrt{T}}\right)-e^{2b_2(\theta-b_1)}N\left(\frac{b(T)-T\theta-2b_2}{\sqrt{T}}\right)\to 1-e^{2b_2(\theta-b_1)}, \tag{3.3}$$

   which is strictly positive. Therefore, the whole limit is infinity since $k>-\frac{\theta^2}{2}$. Note that limit (3.3) is zero when $b_2=0$ or $b_1=\theta$ – we have a discontinuity in these points.

8. Note that the arguments in the normal CDFs tend to $(b_1-\theta)\sqrt{T}$. We derive the desired result using the second statement of Lemma 2.3.

9. The result follows the second statement of Lemma 2.4.

The rest of the limits can be derived in the same manner using formula (3.2) instead (3.1). □

Theorem 3.2. *Let us have another linear function $z(t)=z_1t+z_2$ in addition to the assumptions of Theorem* 3.1*. It has to satisfy the following conditions:*

1. *If $b_2>0$, then $z_1\le b_1$. In addition, if $z_1=b_1$, then $z_2<b_2$.*

2. *If $b_2<0$, then $z_1\ge b_1$. In addition, if $z_1=b_1$, then $z_2>b_2$.*

Under these assumptions, the statements 1–3, 5 *and* 10 *of Theorem* 3.1 *applied to the limit $\lim_{T\to\infty}e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta,B_T>z(T)}\right]$ hold. The statements* 8, 9, 13, *and* 14 *are similar, but the critical value at which the limit changes from zero to infinity is $\frac{z_1^2}{2}-z_1\theta$ instead $\frac{b_1^2}{2}-b_1\theta$. The fourth, sixth, seventh, eleventh, and twelfth cases are devised into the following sub-cases:*

4.1 *If $b_2\neq 0$, $z_1<\theta=b_1$ and $-\frac{\theta}{2}<k$, then*

$$\lim_{T\to\infty}e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta,B_T>z(T)}\right]=\infty.$$

4.2 *If* $b_2 \neq 0,\ z_1 = \theta = b_1$ *and* $-\frac{\theta}{2} < k \leq \frac{b_1^2}{2} - b_1\theta,$ *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = 0.$$

4.3 *If* $b_2 \neq 0,\ z_1 = \theta = b_1$ *and* $\frac{b_1^2}{2} - b_1\theta < k,$ *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = \infty.$$

6.1 *If* $b_2 > 0,\ z_1 < \theta < b_1,$ *and* $k = -\frac{\theta^2}{2},$ *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = 1 - e^{2b_2(\theta-b_1)}.$$

6.2 *If* $b_2 > 0,\ z_1 = \theta < b_1,$ *and* $k = -\frac{\theta^2}{2},$ *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = \frac{1}{2}\left(1 - e^{2b_2(\theta-b_1)}\right).$$

6.3 *If* $b_2 > 0,\ \theta < z_1 \leq b_1,$ *and* $k = -\frac{\theta^2}{2},$ *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = 0.$$

7.1 *If* $b_2 > 0,\ z_1 \leq \theta < b_1,$ *and* $-\frac{\theta}{2} < k,$ *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = \infty.$$

7.2 *If* $b_2 > 0,\ \theta < z_1 \leq b_1$ *and* $-\frac{\theta}{2} < k \leq \frac{z_1^2}{2} - z_1\theta,$ *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = 0.$$

7.3 *If* $b_2 > 0,\ \theta < z_1 \leq b_1$ *and* $\frac{z_1^2}{2} - z_1\theta < k,$ *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = \infty.$$

11.1 *If* $b_2 < 0,\ z_1 > \theta > b_1,$ *and* $k = -\dfrac{\theta^2}{2}$, *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T<z(T)}\right] = 1 - e^{2b_2(\theta - b_1)}.$$

11.2 *If* $b_2 < 0,\ z_1 = \theta > b_1,$ *and* $k = -\dfrac{\theta^2}{2}$, *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T<z(T)}\right] = \frac{1}{2}\left(1 - e^{2b_2(\theta - b_1)}\right).$$

11.3 *If* $b_2 < 0,\ \theta > z_1 \geq b_1,$ *and* $k = -\dfrac{\theta^2}{2}$, *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T<z(T)}\right] = 0.$$

12.1 *If* $b_2 < 0,\ z_1 \geq \theta > b_1,$ *and* $-\dfrac{\theta}{2} < k$, *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = \infty.$$

12.2 *If* $b_2 < 0,\ \theta > z_1 \geq b_1$ *and* $-\dfrac{\theta}{2} < k \leq \dfrac{z_1^2}{2} - z_1\theta$, *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = 0.$$

12.3 *If* $b_2 < 0,\ \theta > z_1 \geq b_1$ *and* $\dfrac{z_1^2}{2} - z_1\theta < k$, *then*

$$\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = \infty.$$

P r o o f. Having in mind the assumptions for the coefficients z_1 and z_2, we prove the statements 1–3 and 5 using similar arguments as in Theorem 3.1.

Let us consider the fourth case. The sub-case 4.1 is obvious because the second part in the limit vanishes when $z_1 < \theta$. The results in cases 4.2 and 4.3 follow inequalities (2.3) and the whole Corollary 2.1.

The difference in the sixth statement arises from the inequalities $\theta < b_1$ and $z_1 \leq b_1$ – thus we need to position z_1 w.r.t. θ.

Let us consider the seventh case. The result for $z_1 \leq \theta$ is obvious. Suppose now that $\theta < z_1 < b_1$ and $z_2 < b_2$. We rewrite expectation (2.1) as

(3.4)
$$\mathbb{E}\left[e^{\theta B_T} I_{B_T > z(T), \zeta > T}\right] = \exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} N\left(\frac{b(T) - T\theta}{\sqrt{T}}\right) - N\left(\frac{z - T\theta}{\sqrt{T}}\right) \\ + e^{2b_2(\theta - b_1)}\left(N\left(\frac{z - T\theta - 2b_2}{\sqrt{T}}\right)\right. \\ \left. - N\left(\frac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right)\right) \end{array}\right]$$

$$= e^{2b_2(\theta - b_1)} \exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} N\left((\theta - b_1)\sqrt{T} + \frac{b_2}{\sqrt{T}}\right) \\ - e^{2b_2(b_1 - \theta)} N\left((\theta - b_1)\sqrt{T} - \frac{b_2}{\sqrt{T}}\right) \\ - N\left((\theta - z_1)\sqrt{T} + \frac{2b_2 - z_2}{\sqrt{T}}\right) \\ + e^{2b_2(b_1 - \theta)} N\left((\theta - z_1)\sqrt{T} - \frac{z_2}{\sqrt{T}}\right) \end{array}\right]$$

If $z_1 = b_1$, then the desired result follows Corollary 2.1. If $z_1 < b_1$ and $-\frac{\theta}{2} < k \leq \frac{z_1^2}{2} - z_1\theta$, then we obtain the limits using the second inequality of Lemma 2.2. If $\frac{z_1^2}{2} - z_1\theta < k$, we apply Lemma 2.5 for $m_1 = \theta - z_1 < 0$, $m_2 = b_2 > 0$, $C = e^{2b_2(b_1 - z_1)} > 1$, and $D = b_2 - z_2 > 0$. If $z_2 > b_2$, we use presentation (2.8).

For cases 8 and 9 we use the original presentation (2.1) instead (3.4) since $b_1 < \theta$. The results can be obtained in a similar to the case 7 way via Corollary 2.1 and Lemma 2.5.

Some symmetrical arguments prove the desired results when $b_1 < 0$. $\square$

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